

# A GEOMETRIC CHARACTERIZATION OF ORIENTATION REVERSING INVOLUTIONS

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ABSTRACT. We give a geometric characterization of compact Riemann surfaces admitting orientation reversing involutions with fixed points. Such surfaces are generally called real surfaces and can be represented by real algebraic curves with non-empty real part. We show that there is a family of disjoint simple closed geodesics that intersect all geodesics of a partition at least twice in uniquely right angles if and only if the involution exists. This implies that a surface is real if and only if there is a pants decomposition of the surface with all Fenchel-Nielsen twist parameters equal to 0 or  $\frac{1}{2}$ .

## 1. INTRODUCTION

A smooth complex projective algebraic curve  $C$  can be represented by a compact Riemann surface  $S$  (i.e. an orientable compact surface with a conformal structure). The curve  $C$  can be described by real polynomial equations if and only if the surface  $S$  admits an orientation reversing involution  $\sigma$  defined by complex conjugation. If the set of real points of  $C$  is not empty then the fixed point set of  $\sigma$  is non-empty or equivalently the field of meromorphic functions of  $(S, \sigma)$  is real (i. e. in such field  $-1$  is not a sum of squares). Hence a surface  $S$  admitting an orientation reversing involution  $\sigma$  with  $\text{Fix}(\sigma) \neq \emptyset$  is called a *real Riemann surface* and  $\sigma$  is a real form on  $S$ .

In this article we deal with one of the fundamental problems in this subject: to decide whether a complex algebraic curve admits a real form, i. e. whether a complex algebraic curve can be described using polynomials with real coefficients. Equivalently the problem is to describe in  $\mathcal{M}_g$  (moduli space of surfaces of genus  $g$ ) the real moduli  $\mathcal{M}_g^{\mathbb{R}}$  whose points are the real Riemann surfaces. The main result determines, in the Teichmüller space of Riemann surfaces of genus  $g$ ,  $\mathcal{T}_g$ , a subspace  $\mathbb{R}\mathcal{T}_g$  that projects on  $\mathcal{M}_g^{\mathbb{R}}$ . We use the classic Fenchel-Nielsen parametrization of  $\mathcal{T}_g$ , namely the parameters are the collection of lengths of geodesics in a pants decomposition and twist parameters which describe how the pairs of pants are pasted together. Hence, the parametrization is not homogeneous in the nature of the parameters. We show that all points of  $\mathcal{M}_g^{\mathbb{R}}$  can be represented by elements of  $\mathcal{T}_g$  where

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all the twist parameters are zero or  $1/2$  (and vice-versa), hence the Fenchel-Nielsen parametrization restricted to  $\mathbb{RT}_g$  produces a parametrization where all parameters are of homogeneous nature. Real moduli has recently been studied by several authors, see for instance [6], [7], and [14], with new and classical applications.

In order to prove the above result we obtain a geometric characterization of the Riemann surfaces admitting an orientation reversing involution by showing that such surfaces are exactly the Riemann surfaces having a pants decomposition with with zero or  $1/2$  twist parameters. The pants decomposition turns out to be not only invariant by the orientation reversing involution but furthermore the involution induces the identity of the graph of such a decomposition. This essential fact is no longer possible if one considers an orientation reversing involution without fixed points, as the number of disjoint simple closed geodesics that are left invariant by such an involution cannot exceed  $g + 1$ . We also mention that the usage of invariant pants decompositions for real Riemann surfaces can be found in Buser and Seppälä's paper [5], but they allow the involution to define a non-trivial combinatorial involution on the graphs of such decompositions.

The main result is Theorem 3.2 where we prove that a Riemann surface is real if and only if there exists a set  $\mathcal{B}$  of disjoint simple closed geodesics that intersect all geodesics of a pants decomposition at least twice at perpendicular angles. There are similar geometrical characterizations for surfaces admitting conformal involutions, obtained by Maskit [13] and Schmutz-Schaller ([16], [17]). Using uniformization groups, there are other characterizations of real Riemann surfaces, see for instance [18], [2], and [10].

Finally, in Section 4, we obtain as a consequence information on the upper bound of the distance between fixed points of an orientation reversing involution and other points of the surface. We also provide an example illustrating this last result.

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## 2. PRELIMINARIES

Our main object of study is a compact Riemann surface  $S$  (i.e. an orientable compact surface with a conformal structure) of genus  $g \geq 2$ . It can be endowed with a hyperbolic metric which we shall denote  $d(\cdot, \cdot)$ . Generally, curves and geodesics will be considered primitive and non-oriented, and will be seen as point sets on  $S$ . Occasionally, we will need to consider surfaces with boundary, and the signature of such a surface will be denoted  $(g, k)$  where  $g$  is the topological genus and  $k$  is the number of simple boundary curves. Unless specified, boundary curves will be considered geodesic. The following propositions concern well known properties of curves on hyperbolic surfaces (i.e. [4], pp. 19-23).

**Proposition 2.1.** *Let  $S$  be a hyperbolic surface. Let  $\alpha, \beta$  be disjoint simple closed geodesics on  $S$ . Let  $c$  be a simple path from  $\alpha$  to  $\beta$ . Then in the free homotopy class of  $c$  with endpoints gliding on  $\alpha$  and  $\beta$ , there exists a unique shortest curve, denoted*

$\mathbb{G}(c)$ , which meets  $\alpha$  and  $\beta$  perpendicularly. Furthermore, if  $\tilde{c}$  is also a simple path from  $\alpha$  to  $\beta$  such that  $c \cap \tilde{c} = \emptyset$ , then either  $\mathbb{G}(c) = \mathbb{G}(\tilde{c})$  or  $\mathbb{G}(c) \cap \mathbb{G}(\tilde{c}) = \emptyset$ .

In the case of simple closed curves, the corresponding proposition is the following. Note that we call a piecewise geodesic boundary curve  $c$  *convex* if for all points  $p$  of  $c$  the interior angle  $\theta_p$  verifies  $\theta_p \leq \pi$ .

**Proposition 2.2.** *Let  $S$  be a compact hyperbolic surface (possibly with piecewise geodesic convex boundary) and let  $c$  be a homotopically non-trivial simple closed curve on  $S$ . Then  $c$  is freely homotopic to a unique simple closed geodesic, denoted  $\mathbb{G}(c)$ . The curve  $\mathbb{G}(c)$  is either contained in  $\partial S$  or  $\mathbb{G}(c) \cap \partial S = \emptyset$ . If  $c$  is a non-smooth boundary component, then  $\mathbb{G}(c)$  and  $c$  bound an embedded annulus (see figure 2).*

A collection  $\mathcal{P}$  of  $3g - 3$  disjoint simple closed geodesics is called a partition or the geodesics of a pants decomposition, and  $S \setminus \mathcal{P}$  is a collection of  $2g - 2$  surfaces of signature  $(0, 3)$ , commonly called pairs of pants or  $Y$ -pieces. Following proposition 2.1, between two distinct boundary geodesics of a  $Y$ -piece  $\mathcal{Y}$ , there is a unique geodesic path perpendicular to the boundary geodesics. The three perpendicular paths defined in this way are disjoint, and by cutting along them one obtains two anticonformal isometric hexagons.

There are different ways to define what are commonly called *twist parameters*. The definition we will use is the following, which is relative to a given partition.

**Definition 2.3.** *Let  $\gamma$  be a geodesic in  $\mathcal{P}$  with a given orientation. Let  $\mathcal{Y}_1 = (\alpha, \beta, \gamma)$  and  $\mathcal{Y}_2 = (\alpha', \beta', \gamma)$  be the two  $Y$ -pieces pasted along  $\gamma$ . Let  $\ell_{\alpha\gamma}$  be the perpendicular between  $\alpha$  and  $\gamma$  on  $\mathcal{Y}_1$ . Let  $\ell_{\alpha'\gamma}$  be the perpendicular between  $\alpha'$  and  $\gamma$  on  $\mathcal{Y}_2$ . Let  $p$  be the intersection point between  $\gamma$  and  $\ell_{\alpha\gamma}$  and let  $q$  be the intersection point between  $\gamma$  and  $\ell_{\alpha'\gamma}$ . Let  $\ell$  be the length of the path on  $\gamma$  from  $p$  to  $q$  following the orientation of  $\gamma$ . The twist parameter  $t_\gamma$  along  $\gamma$  is defined as the quantity  $\frac{\ell}{\gamma}$ .*

The above definition fixes twist parameters in the interval  $[0, 1[$ , and the definition can be extended to a parameter that lies in  $\mathbb{R}$ . Using the extended version of twist parameters and the lengths of geodesics in a partition, one obtains the Fenchel-Nielsen parametrization of the space of marked Riemann surfaces of genus  $g$  ([8], [9], or Teichmüller space of genus  $g$  surfaces, denoted  $\mathcal{T}_g$ . This shows that  $\mathcal{T}_g$  is homeomorphic to  $(\mathbb{R}^{+*})^{3g-3} \times \mathbb{R}^{3g-3}$ . The Fenchel-Nielsen parametrization of Teichmüller space is relative to a graph describing the partition where the graph associated to a pants decomposition has vertices which correspond to  $Y$ -pieces and two vertices are connected if and only if there is a common boundary for the two corresponding  $Y$ -pieces. Restricting the twist parameters to the interval  $[0, 1[$  still ensures us that we have at least one representative (in fact an infinity) of each conformal equivalence class of surfaces of genus  $g$ . The space of all conformal equivalence classes of surfaces of genus  $g$  is called the Moduli space and is denoted  $\mathcal{M}_g$ .

An *involution* is an isometry of the surface onto itself that is of order 2. An involution can either be orientation reversing or not. The following remarks hold for

any orientation reversing involution, although in the sequel we will see that there are fundamental differences between the case where an involution is with or without fixed points.

The following proposition is an extension of what is generally called Harnack's theorem ([19]) and can be found in [12].

**Proposition 2.4.** *If a surface  $S$  admits  $\sigma$ , an orientation reversing involution, then the fixed point set of  $\sigma$  is a set of  $n$  disjoint simple closed geodesics  $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$  with  $n \leq g + 1$ . In the case where the set  $\mathcal{B}$  is separating, then  $S \setminus \mathcal{B}$  consists of two connected components  $S_1$  and  $S_2$  such that  $\partial S_1 = \partial S_2 = \mathcal{B}$  and  $S_2 = \sigma(S_1)$ . If not, then  $\mathcal{B}$  can be completed by either a set  $\alpha$  which consists of one or two simple closed geodesics such that  $\mathcal{B} \cup \alpha$  has the properties described above (with the important difference that  $\alpha$  does not contain any fixed points of  $\sigma$ ). Each of the simple closed geodesics in  $\alpha$  is globally fixed by  $\sigma$ .*

The following proposition concerns any simple closed geodesic fixed by an orientation reversing involution.

**Proposition 2.5.** *Let  $S$  be a surface admitting an orientation reversing involution  $\sigma$ . Let  $\gamma \subset S$  be a simple closed geodesic such that  $\sigma(\gamma) = \gamma$ . If  $\gamma$  does not contain any fixed points of  $\sigma$ , then for all  $p \in \gamma$  the image  $\sigma(p)$  of  $p \in \gamma$  is the point on  $\gamma$  diametrically opposite from  $p$ .*

*Proof.* For any  $p \in \gamma$ , if the points  $p$  and  $\sigma(p)$  are distinct then they separate  $\gamma$  into two geodesic arcs. The image of one of the arcs is either the other arc or is globally fixed. Since  $\sigma$  is an isometry, the mid point of the fixed arc must be fixed by  $\sigma$  and the result follows.  $\square$

### 3. ORIENTATION REVERSING INVOLUTIONS WITH FIXED POINTS

This section will be devoted to a geometric characterization of surfaces admitting an orientation reversing involution with fixed points. We denote by  $\sigma$  an orientation reversing involution with fixed points. Thus the fixed point set of  $\sigma$  is a non-empty set of  $n$  disjoint simple closed geodesics  $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$  with  $n \leq g + 1$ . By cutting  $S$  along  $\mathcal{B}$ , one obtains either one or two surfaces with boundary. The following lemma concerns surfaces with boundary, and will be useful in the sequel.

**Lemma 3.1.** *Let  $S$  be a Riemann surface with non-empty boundary with boundary geodesics  $\beta_1, \dots, \beta_k$ . Let  $c_1, \dots, c_j$  be disjoint simple geodesic paths perpendicular to the boundary of  $S$ . Then  $j \leq 6g - 6 + 3n$  and the set of  $c_i$ s can be completed by  $c_{l+1}, \dots, c_{6g-6+3n}$  such that the new set verifies the same conditions as above.*

*Proof.* The idea of the proof is to show that  $c_1, \dots, c_j$  can be completed into a set that decomposes the surface into hexagons. Once this is obtained, we will show that all further simple geodesic paths perpendicular to boundary must intersect curves in this set, and that the number of simple paths is exactly  $6g - 6 + 3n$ .

The connected components of  $S \setminus \{c_1, \dots, c_j\}$  are orientable surfaces with boundary. All boundary curves are piecewise geodesic and are composed of an even number of simple geodesic paths. Furthermore, an orientation can be given to the boundary curve such that the oriented angles of intersection are always  $\frac{\pi}{2}$ . The boundary geodesics are either trivial (they are the boundary of a geodesic polygon) or are freely homotopic to a simple closed geodesic on  $S$  that is disjoint with the considered boundary.

First let us deal with the case when a boundary curve, say  $b$ , is the boundary of a hyperbolic polygon. The polygon is a convex right-angled  $2i$ -gon with  $i \geq 3$ .

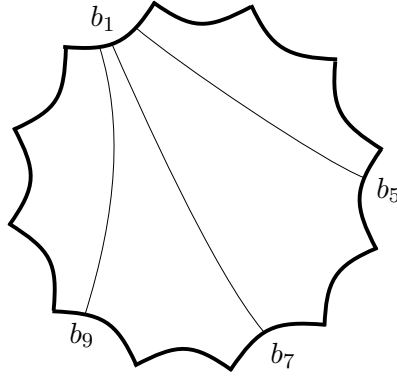


FIGURE 1. The decomposition of a 12-gon

If  $b_1, \dots, b_{2i}$  are the simple geodesic paths that compose  $b$ , then notice that the numbering can be chosen such that for  $j$  odd,  $b_j$  is an element of  $\beta$ , and for  $j$  even,  $b_j$  is a  $c_{j_0}$  for some  $j_0$ . From  $b_1$ , consider the unique geodesic perpendicular paths to each  $b_j$  for  $5 \leq j \leq 2i - 3$  odd. (Put each one of these paths in our collection of  $c_j$ s.) By cutting along these paths, one obtains  $i - 2$  hyperbolic right-angled hexagons as desired.

If  $b$  is not a trivial boundary curve, and  $\mathbb{G}(b)$  the unique simple closed geodesic on  $S$  that is freely homotopic to it, then  $b$  and  $\mathbb{G}(b)$  are the boundary curves of a topological cylinder embedded in  $S$ .

Say  $b_1, \dots, b_{2i}$  are the simple geodesic paths that compose  $b$ . As before, the numbering can be chosen such that for  $j$  odd,  $b_j$  is an element of  $\beta$ , and for  $j$  even,  $b_j$  is a  $c_{j_0}$  for some  $j_0$ .

The next step is to cut hyperbolic hexagons out of the cylinders in order to reduce the boundary of the cylinder to a curve with two geodesic components. Consider the unique geodesic perpendicular path  $c$  from  $\mathbb{G}(b)$  to  $b_1$ . By cutting along  $c$  and

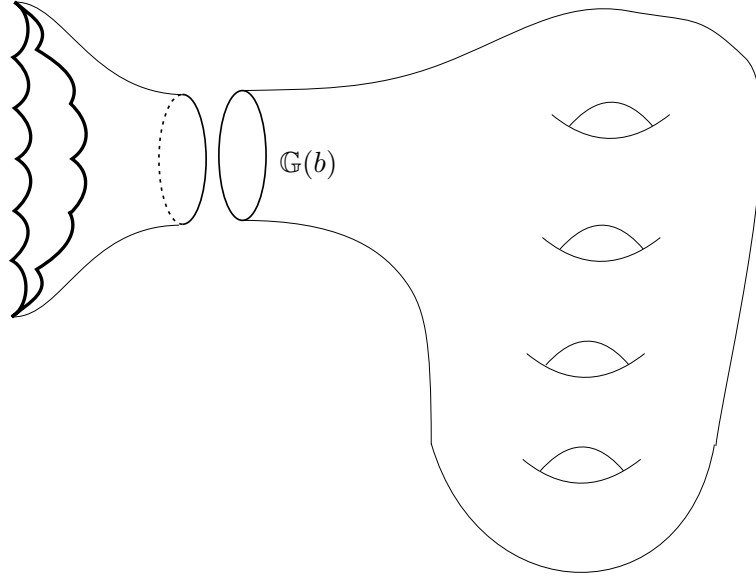
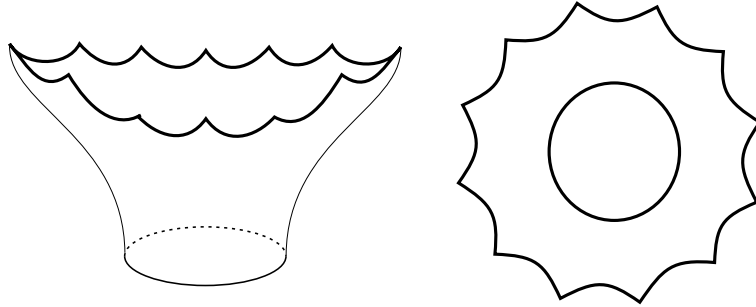


FIGURE 2. The cylinder seen on the surface

FIGURE 3. Two ways of seeing the cylinder enclosed by  $b$  and  $\mathbb{G}(b)$ 

$\mathbb{G}(b)$  one obtains a  $(2i + 4)$ -gon, and following the same procedure as for polygons, one obtains  $i$  hexagons. Now pasting along the path  $c$  we have a cylinder with two boundary components. One of these is a geodesic 2-gon. Notice that in the 2-gon, one of the geodesic paths is an embedded element of  $\beta$ , and the other is a boundary to boundary path.

For all  $b$ , geodesic boundary paths, consider  $\mathbb{G}(b)$ . Cutting along these, one obtains a collection of surfaces with simple geodesic boundaries, and cylinders obtained as above. On a surface with simple geodesic boundaries, consider a perpendicular

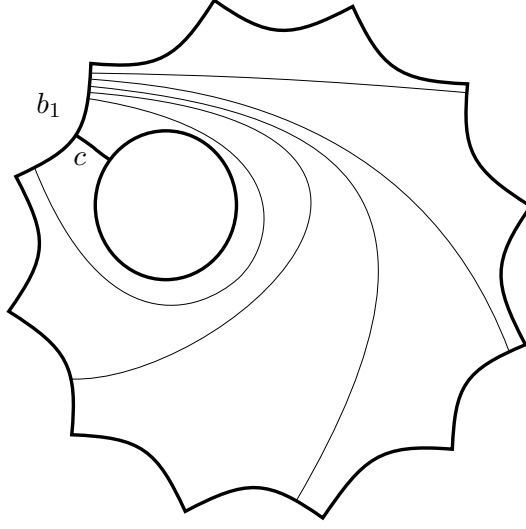


FIGURE 4. The procedure to obtain a piecewise geodesic boundary with two components

geodesic path  $c$  between two boundary geodesics, say  $\gamma$  and  $\delta$ . If  $\gamma$  and  $\delta$  are connected components of  $\beta$ , then the path is admissible. If not, at least one (say  $\gamma$ ) is perpendicular to the boundary geodesic of a cylinder as described above. On this cylinder consider the unique perpendicular geodesic path  $\tilde{c}$  from the embedded element of  $\beta$  to  $\gamma$ . If  $c \cap \gamma = p$  and  $\tilde{c} \cap \gamma = q$ , then consider the (or a) shortest path on  $\gamma$  between  $p$  and  $q$ , say  $\tilde{c}$ . Then  $\mathbb{G}(c \cup \tilde{c} \cup \tilde{c})$  is a simple perpendicular path as desired. Notice that the procedure works even if  $\gamma = \delta$  is a geodesic embedded in a cylinder. After each cut, the boundary curves may have to be reduced to geodesic 2-gon boundary as explained above, or a geodesic polygon may have been cut off.

This procedure can be pursued until the surface has been cut into hexagons. We shall call the completed set of paths along which  $S$  has been cut  $\{c_i\}_{i=1}^{\tilde{n}}$ . Notice that on such a hexagon, if a side is an embedded part of  $\beta$ , then its adjacent sides are two (not necessarily distinct)  $c_i$ s (and vice-versa). Although there are still perpendicular paths between sides on hexagons, they are necessarily between opposite edges, thus from embedded parts of  $\beta$  to a  $c_i$ . This fact proves the maximality of the set  $\{c_i\}$ . We must now count the hexagons. As the hyperbolic area of a right-angled hexagon is  $\pi$ , and the area of  $S$  is  $2\pi(2g - 2 + n)$ , it follows that there are  $2(2g - 2 + n)$  hexagons. Each hexagon has three  $c_i$ s and each path is counted twice. It follows that  $\tilde{n} = 6g - 6 + 3n$  and the lemma is proved.  $\square$

We can now proceed to the main result.

**Theorem 3.2.** *Let  $S$  be a Riemann surface of genus  $g$ . Then  $S$  admits an orientation reversing involution  $\sigma$  with fixed points if and only if there exists a set  $\mathcal{B}$  of*

*disjoint simple closed geodesics and a partition  $\mathcal{P}$  such that all intersections between the geodesics of two sets are perpendicular and such that all geodesics in  $\mathcal{P}$  intersect  $\mathcal{B}$  at least twice. Furthermore,  $\text{Fix}(\sigma) = \mathcal{B}$  and  $\sigma(\gamma) = \gamma$  for all  $\gamma \in \mathcal{P}$ .*

*Proof.* Let us begin by proving the existence of  $\mathcal{B}$  and  $\mathcal{P}$  as above if  $S$  is real. Denote by  $\sigma$  the real involution of  $S$  and by  $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$  the fixed point set of  $\sigma$ . There are two cases to consider. The first case is when  $\mathcal{B}$  separates  $S$  into two surfaces  $S_1$  and  $S_2$  such that  $\sigma(S_1) = S_2$ . Notice that  $g$  and  $n$  are of different parity both  $S_1$  and  $S_2$  are of signature  $(g', n)$  where  $g' = (g - n + 1)/2$ . We can apply lemma 3.1 and on  $S_1$  there exists a set  $c_1, \dots, c_{3g-3}$  of disjoint simple geodesic perpendicular to boundary paths. As all points of  $\mathcal{B} = \partial S_1$  are fixed by  $\sigma$ , it follows that for all  $i \in \{1, \dots, 3g - 3\}$ ,  $\gamma_i = c_i \cup \sigma(c_i)$  is a simple closed geodesic. The set  $\mathcal{P} = \{\gamma_1, \dots, \gamma_{3g-3}\}$  is a partition, as all  $\gamma_i$ s are disjoint. Each  $\gamma_i$  intersects the set  $\mathcal{B}$  twice and in right angles.

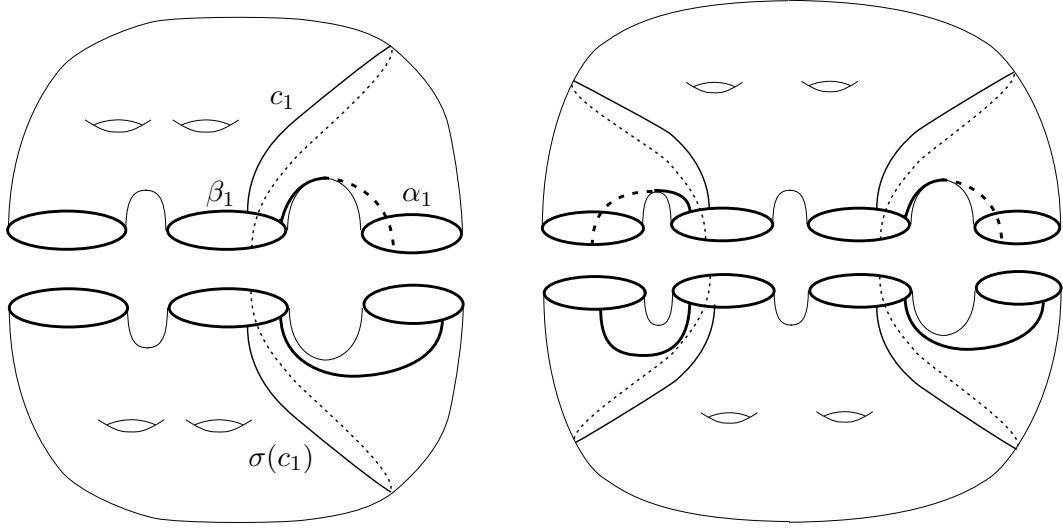
The idea for the case where  $\mathcal{B}$  is not separating is identical, but necessitates a prelude. Consider the set  $\alpha$  defined as in proposition 2.4, which consists of one or two disjoint simple closed geodesics such that  $\sigma(\alpha) = \alpha$  and  $S \setminus \mathcal{B} \setminus \alpha$  has two connected components  $S_1$  and  $S_2$  such that  $\sigma(S_1) = S_2$ . On  $S_1$ , consider a perpendicular simple geodesic path  $c$  oriented from a geodesic in  $\mathcal{B}$ , say  $\beta_1$ , to a geodesic in  $\alpha$ , say  $\alpha_1$ . If  $\alpha_1$  is given an orientation, the piecewise geodesic path  $c\alpha_1c^{-1}$  is freely homotopic to exactly one purely simple geodesic perpendicular path from  $\beta_1$  to  $\beta_1$ , say  $c_1 = \mathbb{G}(c\alpha_1c^{-1})$ . The path  $c_1$  is separating for  $S_1$ , cuts  $\beta_1$  in two, and the surface cut off by  $c_1$  is of signature  $(0, 2)$  with boundaries  $\alpha_1$  and a piecewise geodesic curve consisting of part of  $\beta_1$ , and  $c_1$ . The set  $\gamma_1 = c_1 \cup \sigma(c_1)$  is a separating simple closed geodesic on  $S$ , and cuts off a  $Q$ -piece containing  $\alpha_1$ . If  $\alpha$  contains another simple closed geodesic, say  $\alpha_2$ , then the same procedure can be applied to cut off another  $Q$ -piece containing  $\alpha_2$  with boundary geodesic  $\gamma_2$ , which intersects another element of  $\mathcal{B}$  in two points in a perpendicular fashion, and verifies  $\sigma(\gamma_2) = \gamma_2$ .

The geodesic  $\gamma_1$  (resp. the geodesics  $\gamma_1$  and  $\gamma_2$ ) separate  $S$  into a  $Q$ -piece  $Q_1$ , and a surface  $\tilde{S}$  of signature  $(g - 1, 1)$  (resp. into two  $Q$ -pieces  $Q_1, Q_2$ , and a surface  $\tilde{S}$  of signature  $(g - 2, 2)$ ). The proof of when  $\mathcal{B}$  is separating applies to  $\tilde{S}$ , as the set  $\mathcal{B}|_{\tilde{S}}$  is now separating. We shall proceed to deal with  $Q_1$  (and  $Q_2$  if necessary).

Consider  $Q_1$  (the proof for  $Q_2$  is identical if necessary). The geodesic  $\gamma_1$  is exactly the boundary of  $Q_1$ , and by construction,  $\beta_1|_{Q_1} = h$  is a non-trivial simple geodesic perpendicular path from  $\gamma_1$  to  $\gamma_1$ . Cutting along  $h$ , one obtains a surface  $C$  of signature  $(0, 2)$ , with piecewise geodesic boundary consisting of  $c_1 \cup h_1$  and  $\sigma(c_1) \cup h_2$  where  $h_1$  and  $h_2$  are the two copies of  $h$  on  $C$ . Notice that  $h_1 = \sigma(h_2)$  and that  $\alpha_1$  is an interior separating geodesic of  $C$ .

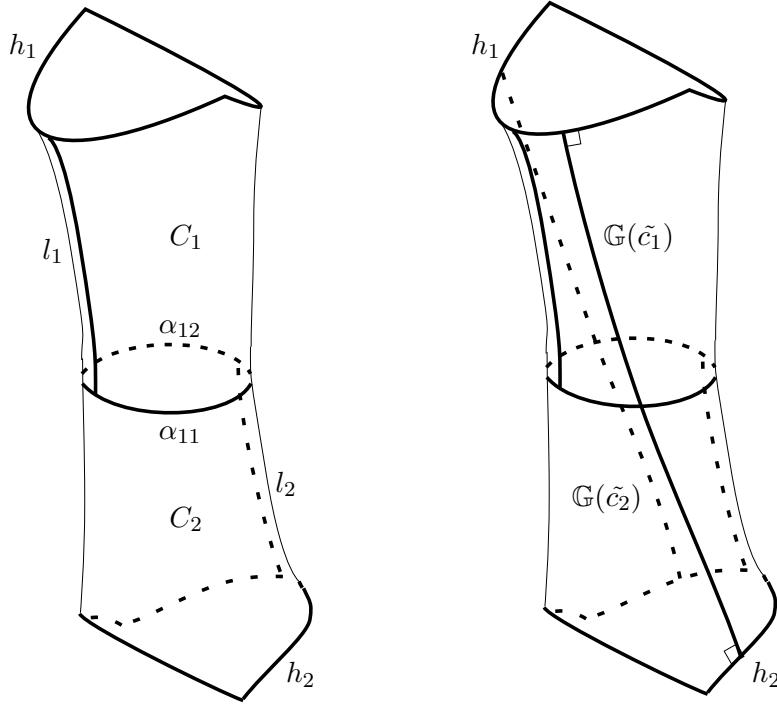
Denote by  $C_1$  (resp.  $C_2$ ) the connected component of  $C \setminus \alpha_1$  with boundary  $c_1 \cup h_1$  (resp. the connected component of  $C \setminus \alpha_1$  with boundary  $\sigma(c_1) \cup h_2$ ). Notice that  $\sigma(C_1) = C_2$ . The shortest path  $l_1$  on  $C_1$  between  $\alpha_1$  and  $h_1$  is the only simple geodesic perpendicular path to both  $\alpha_1$  and  $h_1$ . Denote by  $l_2$  the corresponding path on  $C_2$  (between  $h_2$  and  $\alpha_1$ ). Because both  $l_1$  and  $l_2$  are unique on  $C_1$  and  $C_2$ , it



FIGURE 5. The two possible cases if  $\beta$  is non-separating

follows that  $\sigma(l_1) = l_2$ . Denote by  $p_1$  and  $p_2$  the points  $l_1 \cap \alpha_1$  and  $l_2 \cap \alpha_1$ . Cutting along  $\alpha$  from  $p_1$  to  $p_2$ , one obtains two equal length geodesic arcs (as  $p_2 = \sigma(p_1)$ ) which we shall denote by  $\alpha_{11}$  and  $\alpha_{12}$ . Now consider the two homotopically distinct non-oriented paths  $l_1\alpha_{11}l_2$  (resp.  $l_1\alpha_{12}l_2$ ) and denote them by  $\tilde{c}_1$  and  $\tilde{c}_2$ . It is easy to see that the geodesics in their free homotopy class are disjoint, simple, perpendicular to both  $h_1$  and  $h_2$  and that  $\tilde{c}_2 = \sigma(\tilde{c}_1)$ . On  $S$  they are both perpendicular paths from  $\beta$  to  $\beta$  and it follows that on  $\tilde{c}_1 \cup \tilde{c}_2$  is a simple closed geodesic that intersects  $\beta_1$  twice at right angles, and does not further intersect any element of  $\mathcal{B}$ .

We shall now prove the reciprocal. Consider a set  $\beta$  and a partition  $\mathcal{P}$  that intersects  $\beta$  as in the hypotheses. As each geodesic in  $\mathcal{P}$  intersects  $\beta$  at least twice, then for a pair of pants  $\mathcal{Y}$  in the underlying pants decomposition, the three perpendicular paths from boundary to boundary are subsets of  $\beta$ . To prove this, consider the boundary geodesics of  $\mathcal{Y}$ , say  $\gamma_1, \gamma_2, \gamma_3$ . The conditions impose that  $\beta|_{\mathcal{Y}}$  must consist of simple perpendicular paths between the boundary geodesics. Thus, the connected components of  $\beta|_{\mathcal{Y}}$  are either one of the three perpendiculars mentioned above (type 1), or simple perpendicular paths that whose end points lie on a same boundary geodesic (type 2). If  $\beta|_{\mathcal{Y}}$  consists only of paths of type 1, then as each boundary geodesic intersects at least two of them, the three perpendicular paths of type 1 are contained in  $\beta|_{\mathcal{Y}}$ . If  $\beta|_{\mathcal{Y}}$  contains an element of type 2, then suppose that it intersects  $\gamma_1$ . In this case, then  $\gamma_2$  and  $\gamma_3$  can only intersect one perpendicular path (both of type 1) and this is a contradiction.

FIGURE 6.  $C$  and certain curves

It follows that every pair of pants  $\mathcal{Y}_i \in S \setminus \mathcal{P}$  is divided into two anticonformal isometric hexagons, say  $H_i$  and  $\tilde{H}_i$ , by  $\beta|_{\mathcal{Y}_i}$ . For every  $\mathcal{Y}_i \in \mathcal{P}$ , consider the local orientation reversing involution  $\sigma_i$  that takes a point on  $H_i$  to its corresponding point  $\tilde{H}_i$ . The involutions  $\sigma_i$  can be extended to act on  $\mathcal{P}$ . This gives a unique involution  $\sigma_{\mathcal{P}}$  that acts on the geodesics of  $\mathcal{P}$ . From the set  $\sigma_i$ s and  $\sigma_{\mathcal{P}}$ , we obtain an application  $\sigma$  on  $S$  defined by  $\sigma(p) = \sigma_i(p)$  if  $p \in \mathcal{Y}_i$ . It is straightforward to see that  $\sigma$  is an orientation reversing involution that verifies  $\text{Fix}(\sigma) = \beta$  and  $\sigma(\gamma) = \gamma$  for all  $\gamma \in \mathcal{P}$ .  $\square$

This result can be seen in function of Fenchel-Nielsen parameters.

**Corollary 3.3.** *Let  $S$  be a Riemann surface of genus  $g$ . Then  $S$  admits an orientation reversing involution  $\sigma$  with fixed points if and only if there exists a pants decomposition such that the Fenchel-Nielsen twist parameters are all equal to 0 or  $\frac{1}{2}$ .*

*Proof.* If such an involution exists, then consider the partition  $\mathcal{P}$  and set of geodesics  $\mathcal{B}$  guaranteed by the previous theorem. Consider the set of  $Y$ -pieces  $\{\mathcal{Y}_i\}$  that form  $S \setminus \mathcal{P}$ . For all  $i \in 1, \dots, 2g - 2$ , the set  $\mathcal{B}|_{\mathcal{Y}_i}$  is the set of the perpendicular paths of type 1 on  $\mathcal{Y}_i$ . Hence the only possibility is that the twist parameters lie in the set  $\{0, \frac{1}{2}\}$  if  $\mathcal{B}$  is to consist of simple closed geodesics. Reciprocally, if there exists a

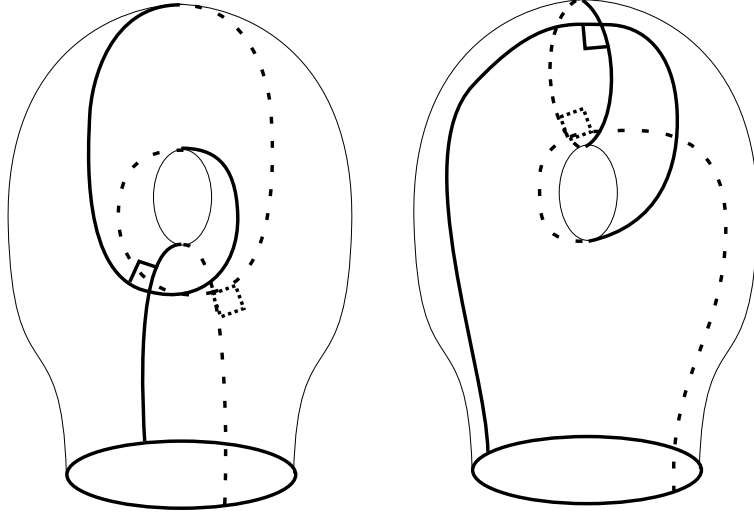


FIGURE 7. The partition geodesic obtained seen two different ways

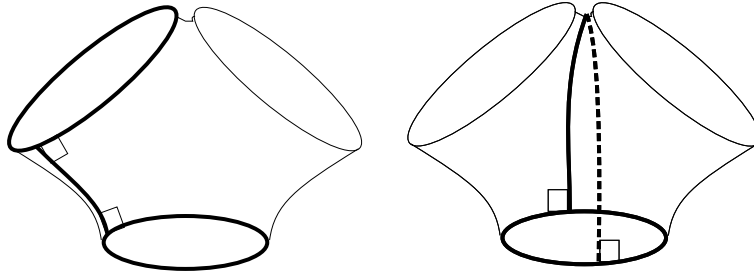


FIGURE 8. Perpendicular paths of type 1 and 2

pants decomposition with twist parameters 0 and  $\frac{1}{2}$ , then the perpendicular simple paths between distinct boundary geodesics of the pairs of pants, form a collection of simple closed geodesics which are disjoint and intersect all geodesics in the underlying partition exactly twice.  $\square$

**Remark 3.4.** *The involution acts as the identity on the graph induced by the pants decomposition described above.*

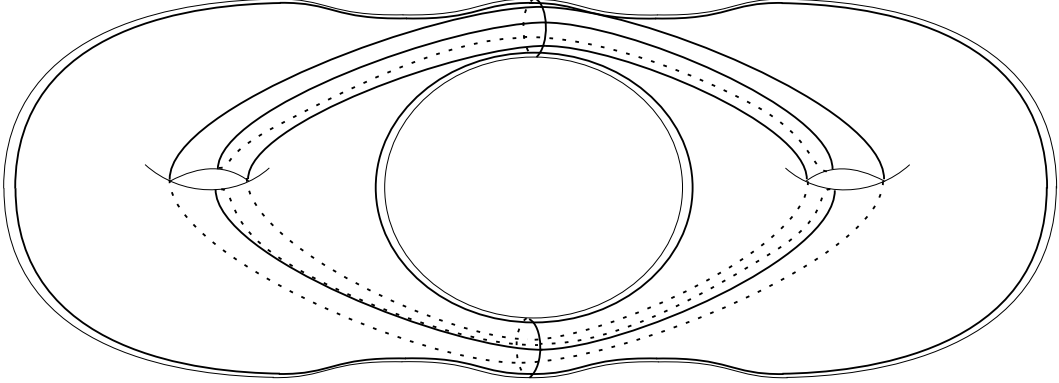


FIGURE 9. A genus 3 surface admitting an involution with two fixed geodesics

#### 4. FURTHER CONSEQUENCES OF THE GEOMETRIC CHARACTERIZATION

One of the main consequences of theorem 3.2 is that it gives a very precise image of surfaces which allow such involutions. This vision for instance allows the following proposition which concerns the distance between fixed points of an orientation reversing involution and other points of the surface.

**Proposition 4.1.** *The following inequality is always true:*

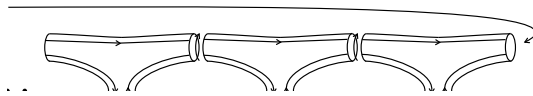
$$(1) \quad \max_{p \in S} d_S(\text{Fix}(\sigma), p) > \frac{\ln 3}{2}.$$

*Reciprocally, for any  $\varepsilon > 0$  and any  $g \geq 2$ , there exists a surface  $S_\varepsilon$  of genus  $g$  with an orientation reversing involution with fixed points (of any given type) such that*

$$(2) \quad \max_{p \in S_\varepsilon} d_{S_\varepsilon}(\text{Fix}(\sigma_\varepsilon), p) < \frac{\ln 3}{2} + \varepsilon.$$

*Proof.* In [15], the existence of a disk of radius  $\frac{\ln 3}{2}$  on the complement of a partition is shown. As the geodesics composing the fixed point set of  $\sigma$  can be completed into a partition, it follows that on  $S \setminus \text{Fix}(\sigma)$ , there is a disk of radius  $\frac{\ln 3}{2}$  and equation 1 follows. For equation 2, it suffices to construct an example. For a given topological type of involution  $\sigma$ , theorem 3.2 ensures us of the existence of a partition  $\mathcal{P}$  such that all perpendicular paths between boundary geodesics of the underlying  $Y$ -pieces are elements of  $\text{Fix}(\sigma)$ . We shall consider the lengths of the geodesics in  $\mathcal{P}$  as free parameters to describe the surface  $S_\varepsilon$  without modifying the twist parameters. Using the methods in [15], for any  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  such that if one chooses all geodesics in  $\mathcal{P}$  of length shorter than  $C_\varepsilon$ , one obtains a surface  $S_\varepsilon$  without any disks of radius  $\frac{\ln 3}{2}$  embedded in  $S_\varepsilon \setminus \mathcal{P} \setminus \text{Fix}(\sigma)$ . This proves equation 2. For more clarity see the example that follows the proof. □

*Example:* In order to illustrate proposition 4.1, consider the following example.

FIGURE 10. An example in genus  $g$ 

Notice that the twist parameters of certain geodesics have been left free. If, for instance, all twist parameters are considered to be  $\frac{1}{2}$ , then we have an example where  $\text{Fix}(\sigma)$  is a single simple closed geodesic. If all twist parameters are equal to 0, then  $\text{Fix}(\sigma)$  consists of  $g + 1$  simple closed geodesics. The twist parameters can be chosen in order to create an example with any topological type of  $\sigma$ .

The fixed point geodesics of  $\sigma$  can be chosen as short as wanted. Hence, the collar theorem (i.e. [3],[11]) implies that for any constant  $C$ , there exists a surface  $S_C$  admitting an orientation reversing involution  $\sigma_C$  with fixed points such that

$$\max_{p \in S_C} d_{S_C}(\text{Fix}(\sigma_C), p) > C.$$

In corollary 3.3, we show the existence of a partition  $\mathcal{P}$  such that each simple closed geodesic is left invariant by  $\sigma$ . In [5], one of the main results concerns partitions of surfaces with an orientation reversing involution. It is shown that a partition  $\tilde{\mathcal{P}}$  can be chosen such that  $\sigma(\tilde{\mathcal{P}}) = \tilde{\mathcal{P}}$ , and that  $\max_{\gamma \in \tilde{\mathcal{P}}} \ell(\gamma) \leq 21g$  where  $g$  is the genus. The existence of such a constant for arbitrary Riemann surfaces of same genus was originally proved in [1]. However, although the set  $\tilde{\mathcal{P}}$  is globally fixed by  $\sigma$ , each geodesic in  $\tilde{\mathcal{P}}$  is not necessarily fixed. The graph of the underlying pants decomposition is invariant, but the involution does not necessarily act as the identity on it. Using the collar theorem as above, it is easy to see that we can not find a bound on the lengths of the geodesics in  $\mathcal{P}$  in function of the genus  $g$ , as was done in [5]. If however, one imposes a lower bound on the lengths of geodesics in  $\text{Fix}(\sigma)$ , then using the methods exposed in [5], for given genus, a partition  $\mathcal{P}$  that verifies the conditions of corollary 3.3, can be chosen with bounded length.

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